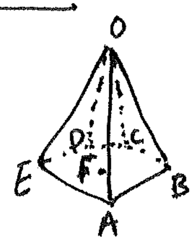
Answermodel Exam Symmetry in Physics April 6, 2020

Exercise 1.

(a)



symmetry transformations:

rotations around axis through O and center of base F
over angles of 0°, 72°, 144°, 216°, 288°

reflections in plane spanned by O, F and any corner (A, B, C, D, E.)

10 in total.

(b) c = rotation over 72°
b = reflection in plane spanned by O, F & A. } bc = reflection in plane spanned by O, F, C

then $c^5 = e$, $b^2 = e$, $(bc)^2 = e$

hence gp {b, c} with $b^2 = c^5 = (bc)^2 = e$ which is D_5.

cycle notation $c = (12345)$, $b = (25)(34)$ } $bc = (15)(24)$ same conclusion.

(c) rotations form 3 classes: $0° = (e)$

$72° = (c) = \{c, c^4\}$ ← related by a reflection

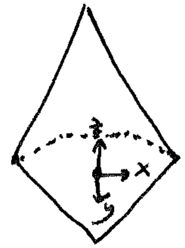
$144° = (c^2) = \{c^2, c^3\}$ ← "

reflection form 1 class $(b) = \{b, bc, bc^2, bc^3, bc^4\}$ all related by rotations: c or c^2

(d) 4 classes = 4 irreps

$$\sum_{i=1}^{4} n_i^2 = 10 \Rightarrow n_1 = n_2 = 1, \; n_3 = n_4 = 2$$ unique solution

(e) D^V using basis



$$D^V(c) = \begin{pmatrix} \cos 72^\circ & \sin 72^\circ & 0 \\ -\sin 72^\circ & \cos 72^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^V(b) = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

2D rep: the upper 2×2 submatrices of D^V:

$$D^{(3)}(c) = \begin{pmatrix} \cos 72^\circ & \sin 72^\circ \\ -\sin 72^\circ & \cos 72^\circ \end{pmatrix} \qquad D^{(3)}(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

irrep: 1) trace:

$$\left.\begin{aligned} \chi(c) &= 2\cos 72^\circ \equiv 2x \\ \chi(c^2) &= 2\cos 144^\circ = 2y \\ \chi(b) &= 0. \end{aligned}\right\} \qquad 2^2 + (2x)^2\cdot 2 + (2y)^2\cdot 2 = 4 + 6 = 10$$

or 2) subgp of $O(2)$, 2D rep is irrep, cannot be diagonalized at same time. (unitary rep, so always fully reducible)

or 3) only matrices that commute with all of them is proportional to $\mathbb{1}_{2\times2}$.

(f) 4 classes = 4 irreps

C_{5v}	(e)	(c)	(c^2)	(b)
$D^{(1)}$	1	1	1	1
$D^{(2)}$	1	1	1	-1
$D^{(3)}$	2	$2x$	$2y$	0
$D^{(4)}$	2	$2y$	$2x$	0

$D^{(2)}$: ← $\chi(c)^5 = 1$ & $\chi(bc) = \chi(b)\chi(c)$ & $\chi(b)^2 = 1$

exercise(e) → $D^{(3)}$

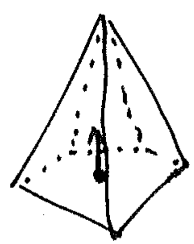
$D^{(4)}$: ← orthogonality or regular rep: $2\cdot(1^2+1^2+(2x)^2+(2y)^2) = 10$

(g) $D^{v} \longrightarrow \chi^{v} = (3,\ 1+2x,\ 1+2y,\ 1)$

$= \chi^{(1)} \oplus \chi^{(3)} \quad \rightarrow \quad D^{v} \sim D^{(1)} \oplus D^{(3)}$

↑ yes, allows for edm in principle, invariant vector direction.

Also allowed: physical picture



invariant vector points in $\hat{z}$ direction.

2a) Symmetry transformations : 1) all rotations in plane of loop.
(forms subgp $SO(2)$)

2) reflection in plane of loop; since axial-vectors orthogonal to the plane will stay invariant under this trafo.

Note no ~~plane~~ reflection in the plane containing the N-S axis will be a symmetry as it would flip axial vectors.

2b) $G_{loop} \neq O(1)$ since G_{loop} is Abelian & $O(2)$ is non-Abelian

2c) $G_{loop} \neq U(1) \cong SO(2)$ since $U(1)$ contains no reflection.

In fact, $G_{loop} \cong SO(2) \times \mathbb{Z}_2$

2d) E.g. nontrivial 1-dim complex irrep of $SO(2)$: $D = (e^{i\varphi})$ $\varphi \in [0, 2\pi]$

If one trivially represents the reflection, then this forms a rep of G_{loop} as well, and since 1D rep it is an irrep

3a) L_z acting on $\begin{pmatrix} |1\,1\rangle \\ |1\,0\rangle \\ |1\,-1\rangle \end{pmatrix}$: $\hbar \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{pmatrix}$

3b) $D^{(\ell=1)}_{m'm}(\theta) = \langle 1\ m' |\ \underbrace{O(\theta, \hat{n})}_{\exp\left(\frac{i}{\hbar}\theta\, \hat{n}\cdot\vec{L}\right)}\ | 1\ m\rangle$ for $\hat{n} = z$

$= \exp\left(\frac{i}{\hbar}\theta L_z\right)$

$$D^{(\ell=1)}(\theta) = \begin{pmatrix} e^{i\theta} & & 0 \\ & 1 & \\ 0 & & e^{-i\theta} \end{pmatrix} \qquad \theta \in [0, 2\pi].$$

3c) $D^{(\ell=1)}(\theta)$ is not itself an $SO(3)$ element, which is defined through the vector rep D^v

But these reps are equivalent as can be seen from their characters: $\mathrm{Tr}\, D^{(\ell=1)} = 1 + e^{i\theta} + e^{-i\theta}$

$= 1 + 2\cos\theta$

$$\mathrm{Tr}\, D^v = \mathrm{Tr}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 + 2\cos\theta$$

equal. hence equivalent

3d) $D^{(\ell=1)} \notin SO(3)$ as ~~the~~ $D^{(\ell=1)T} D^{(\ell=1)} \neq \mathbb{1}_{3\times 3}$

it is not an orthogonal matrix

But matrices representing elements of $SO(3)$ don't need to be in $SO(3)$ themselves, they just have to follow the group multiplication, i.o.w. $D^{(\ell=1)}(R^T) = D^{(\ell=1)\dagger}(R)$

$D^{(\ell=1)}$ differs by a complex basis transformation from D^v that is $\in SO(3)$.